Hosoya index and Fibonacci numbers

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Abstract. Let $G = (V, E)$ be a simple graph. The Hosoya index $Z(G)$ of $G$ is defined as the total number of edge independent sets of $G$. Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion. In this paper, we investigate the relationships between Hosoya index and Fibonacci numbers. Also we consider Fibonacci cubes and study some of its parameters which is related to Fibonacci numbers.

Keywords: Hosoya index; Fibonacci number; Fibonacci cube.

1. Introduction

Let $G = (V, E)$ be a simple graph of order $n$ and size $m$. An $r$-matching of $G$ is a set of $r$ edges of $G$ which no two of them have common vertex. The maximum number of edges in a matching of a graph $G$ is called the matching number of $G$ and denoted by $\alpha'(G)$.

The Hosoya index $Z(G)$ of a graph $G$ is defined as the total number of its matchings [6]. If $m(G, k)$ denotes the number of its $k$-matchings, then

$$Z(G) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} m(G, k).$$

The Hosoya index has been studied intensively in the literature [1, 4, 9, 13, 15, 16].

For $v \in V(G)$, we denote by $G - v$ the graph obtained from $G$ by deleting the vertex $v$ together with their incident edges. For $e \in E(G)$, we denote by $G - e$ the graph obtained from $G$ by removing the edge $e$. Let $\deg(v)$ denotes the vertex degree of $v$. We denote by $P_n, S_n$ and $C_n$ the path, the star and the cycle on $n$ vertices, respectively.
The corona of two graphs $G_1$ and $G_2$, as defined by Frucht and Harary in [3], is the graph $G = G_1 \circ G_2$ formed from one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$, where the $i$th vertex of $G_1$ is adjacent to every vertex in the $i$th copy of $G_2$. The corona $G \circ K_1$, in particular, is the graph constructed from a copy of $G$, where for each vertex $v \in V(G)$, a new vertex $v'$ and a pendant edge $vv'$ are added. The join of two graphs $G_1$ and $G_2$, denoted by $G_1 \vee G_2$, is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup \{uv | u \in V(G_1) \text{ and } v \in V(G_2)\}$.

Fibonacci numbers are terms of the sequence defined in a quite simple recursive fashion. We define Fibonacci numbers:

$$F_0 = 0, F_1 = 1,$$

and for $n \geq 2$,

$$F_n = F_{n-1} + F_{n-2}.$$  

In section 2 we consider graphs with specific constructions and study their Hosoya index. In Section 3 we consider Fibonacci cubes and study some of its parameters which is related to Fibonacci numbers.

2. HOUSOYA INDEX OF CERTAIN GRAPHS

In this section we compute the Hosoya index for some certain graphs. We state the following theorem:

**Theorem 1.** ([5])

1. Let $e = uv$ be an edge of a graph $G$. Then

$$Z(G) = Z(G - e) + Z(G - \{u, v\}).$$

2. Let $v$ be a vertex of a graph $G$. Then

$$Z(G) = Z(G - v) + \sum_{uv \in E(G)} Z(G - uv),$$

where the summation extends over all vertices adjacent to $v$.

3. If $G_1, G_2, \ldots, G_k$ are connected components of $G$, then $Z(G) = \prod_{i=1}^{k} Z(G_i)$.

Here we consider some kind of graphs and obtain their Hosoya indices.

Let $P_{m+1}$ be a path with vertices labeled $y_0, y_1, \ldots, y_m$, for $m \geq 0$ and let $G$ be any graph. Denote by $G_{y_0}(m)$ (or simply $G(m)$, if there is no likelihood of confusion) a graph obtained from $G$ by identifying the vertex $v_0$ of $G$ with an end vertex $y_0$ of $P_{m+1}$ (see Figure 1). For example, if $G$ is a path $P_2$, then $G(m) = P_2(m)$ is the path $P_{m+2}$.

Let $P_m$ be a path with vertices labeled $y_1, \ldots, y_m$ and let $a, b$ be two specific vertices of a graph $G$ (note that may be $a = b$). Denote by $G_{y_1}(m)$ (or simply $G(m)$, if there is no likelihood of confusion) a graph obtained from $G$ and $P_m$ by identifying vertices $a$ and $y_1$ and also $b$ and $y_m$. See Figure 1. Through-out our discussion these two vertices $a$ and $b$ are fixed.
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By Theorem 1, we have the following theorem for Hosoya indices of graphs $G(m), G'(m)$ and $G_1(m)G_2$:

**Theorem 2.**

1. The Hosoya index of graph $G(m)$ satisfy
   \[ Z(G(m)) = Z(G(m-1)) + Z(G(m-2)). \]
2. The Hosoya index of graph $G_1(m)G_2$ satisfy
   \[ Z(G_1(m)G_2) = Z(G_1(1))Z(G_2(m-1)) + Z(G_1)Z(G_2(m-2)). \]
3. The Hosoya index of graph $G'(m)$ satisfy
   \[ Z(G'(m)) = Z(G(1)(m-1)) + Z(G(m-2)). \]

As a consequence of Theorem 2, we have the following corollary:

**Corollary 3.**

1. Let $P_n$ be a path with $n$ vertices. Then for every $n \geq 2$, $Z(P_n) = F_{n+1}$, with $Z(P_0) = 0$, $Z(P_1) = 1$.
2. Let $C_n$ be a cycle of order $n$, then
   \[ Z(C_n) = F_{n-1} + F_{n+1}. \]

**Proof.**

1. Using Theorem 2 (i) for $G = K_1$ we have the result.
2. It suffices to use Theorem 2 (iii) for $G = K_2$.

Let $L_{n,k}(s,t)$ be the set of all unicyclic graphs as shown in Figure 2. We assume that $u_0$ and $v_0$ are adjacent in $L_{n,k}$ and $n$ is the order of graph, i.e. $s + t + k = n$. Obviously these graphs are of the form $C_k(t)(s)$.

![Figure 1](image1.png)

**Figure 1.** Graphs $G(m), G'(m)$ and $G_1(m)G_2$, respectively.

![Figure 2](image2.png)

**Figure 2.** Graph $L_{n,k}(s,t)$.
We have the following theorem for Hosoya indices of graphs $L_{n,k}(s,t)$.

**Theorem 4.**

$$Z(L_{n,k}(s,t)) = F_{n+1} + F_{k-1}F_{n-k+1}.$$  

**Proof.** By Theorem 1 we have

$$Z(L_{n,k}(s,t)) = Z(L_{n,k}(s,t) - v_0v_1) + Z(L_{n,k}(s,t) - v_0 - v_1)$$

$$= Z(P_n)Z(L_{n,x,k}) + Z(P_{s+1})Z(P_{n-s-1})$$

$$= F_{s+1}(F_{n-s+1} + F_{k-1}F_{n-s-k+1}) + F_{s}F_{n-s}$$

$$= F_{s+1}F_{n-s+1} + F_{s}F_{n-s} + F_{k-1}F_{s+1}F_{n-s-k+1}$$

$$= F_{n+1} + F_{k-1}F_{n-k+1}.$$

Here we consider the corona of $P_n$ and $C_n$ with $K_1$. We denote $P_n \circ K_1$ and $C_n \circ K_1$ simply by $P_n^*$ and $C_n^*$, respectively. See Figure 3. We denote the graph obtained from $G_n^*$ by deleting the vertex labeled $2n$ as $G_n^* - \{2n\}$. We have the following theorem:

![Figure 3. Labeled centipede $P_n^*$.](image)

**Theorem 5.**

1. For every $n \geq 3$, $Z(P_n^*) = 2Z(P_{n-1}^*) + Z(P_{n-2}^*)$, $Z(P_1^*) = F_3 = 2$, $Z(P_2^*) = F_5 = 5$.

2. For every $n \geq 3$, $Z(C_n^*) = 2Z(P_{n-1}^*) + 2Z(P_{n-2}^*)$, $Z(P_1^*) = F_3 = 2$, $Z(P_2^*) = F_5 = 5$.

**Proof.**

1. By Theorem 2 (i) we have $Z(P_n^*) = Z(P_n^* - \{2n\}) + Z(P_{n-1}^*)$. Also for graph $P_n^* - \{2n\}$ we have $Z(P_n^* - \{2n\}) = Z(P_{n-2}^*) + Z(P_{n-1}^*)$. By these two equations we have $Z(P_n^*) = 2Z(P_{n-1}^*) + Z(P_{n-2}^*)$.

2. By Theorem 2 (i) we have $Z(C_n^*) = Z(P_n^*) + Z(P_{n-2}^*)$ Now we have the result by part (i).

Now we shall extend previous result to graphs of the form $P_n \circ K_i$ and $C_n \circ K_i$, where $i \geq 1$.

**Theorem 6.**

1. For every $n \geq 3$, $Z(P_n \circ K_i) = (i + 1)Z(P_{n-1} \circ K_i) + Z(P_{n-2} \circ K_i)$. 

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2. For every \( n \geq 3 \), \( Z(C_n \circ K_i) = (i + 1)Z(P_{n-1} \circ K_i) + 2Z(P_{n-2} \circ K_i) \).

3. On the parameters of Fibonacci cubes

As we have seen, the Hosoya index of some graphs are relates to Fibonacci numbers. In this section we shall consider Fibonacci cube and study the relationship between its parameters with Fibonacci numbers.

Let \( B = \{0,1\} \) and for \( n \geq 1 \) set \( B_n = \{b_1b_2...b_n | b_i \in B, 1 \leq i \leq n\} \). The \( n \)-cube \( Q_n \) is the graph defined on the vertex set \( B_n \), vertices \( b_1b_2...b_n \) and \( b'_1b'_2...b'_n \) being adjacent if \( b_i \neq b'_i \) holds for exactly one \( i \in \{1,...,n\} \). Clearly, \( |V(Q_n)| = 2^n \). To obtain additional graphs (or networks) with similar properties as hypercubes, but on vertex sets whose order is not a power of two, Hsu [7] (see also [8]) introduced Fibonacci cubes as follows.

For \( n \geq l \) let \( F_n = \{b_1b_2...b_n \in B_n | b_i b_{i+1} = 0,1 \leq i \leq n-1\} \). The Fibonacci cube \( \Gamma_n \), \( n \geq 1 \), has \( F_n \) as the vertex set, two vertices being adjacent if they differ in exactly one coordinate. In other words, \( \Gamma_n \) is the graph obtained from \( Q_n \) by removing all vertices that contain at least two consecutive 1s. The Fibonacci cube \( \Gamma_5 \) is shown in Figure 4. It is easy to see that \(|V(\Gamma_n)| = F_{n+2}\). For more information on Fibonacci cubes refer to [12].

Fibonacci cubes were introduced as interconnection networks [7] and later studied from many aspects, see the recent survey [11]. From our point of view it is important that Fibonacci cubes also play a role in mathematical chemistry. Fibonacci cubes are precisely the resonance graphs of fibonacenes which in turn form an important class of hexagonal chains [10].

![Figure 4. 5-dimensional Fibonacci cube \( \Gamma_5 \).](image)

A less direct representation of Fibonacci cubes appeared in theoretical chemistry. The resonance graph or the \( Z \)-transformation graph of a hexagonal graph
$H$ has perfect matchings as vertices, two vertices being adjacent if they differ on exactly one 6-cycle and on this cycle their symmetric difference is the whole cycle; see [17] and references therein, as well as [18] for a generalization of this concept to all plane bipartite graphs. A fibonacene is a hexagonal chain in which no three hexagons are linearly attached. These concepts lead to the following representation of Fibonacci cubes:

**Theorem 7.** ([10]) Let $G$ be a fibonacene with $n$ hexagons. Then the resonance graph of $G$ is isomorphic to $\Gamma_n$.

An independent set of a graph $G$ is a set of vertices where no two vertices are adjacent. The independence number is the size of a maximum independent set in the graph and denoted by $\alpha$.

**Lemma 8** ([2]) $\Gamma_n$ has a Hamiltonian path for any $n \geq 0$.

The following theorem give us the independence number of $\Gamma_n$.

**Theorem 9.** For any $n \geq 0$, $\alpha(\Gamma_n) = \left\lceil \frac{F_{n+2}}{2} \right\rceil$.

**Proof:** Since $\Gamma_n$ has a Hamiltonian path, $\alpha(\Gamma_n) \leq \left\lfloor \frac{F_{n+2}}{2} \right\rfloor$. On the other hand, if $X$ and $Y$ are the bipartition of $\Gamma_n$, then $\alpha(\Gamma_n) \geq \left\lceil \frac{F_{n+2}}{2} \right\rceil$. Therefore we have the result.

The following theorem is about the number of edges and Wiener index of Fibonacci cube.

**Theorem 10.** ([11])

1. $|E(\Gamma_n)| = \frac{nF_{n+1} + 2(n+1)F_n}{5}$.
2. For any $n \geq 0$, $W(\Gamma_n) = \sum_{i=1}^{n} F_i F_{n-i+1} F_{n-i+2}$.

A set $S \subseteq V$ is a dominating set if every vertex in $V \setminus S$ is adjacent to at least one vertex in $S$. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set in $G$. Pike and Zhou [14] proved the following lower bound for domination number of Fibonacci cubes.

**Theorem 11.** For any $n \geq 4$, $\gamma(F_n) \geq \left\lceil \frac{F_{n+2} - 3}{n-2} \right\rceil$.

Ordered Hosoya polynomial is the counting polynomial of the distances among ordered pairs of vertices:

$$H(G,x) = \sum_{(u,v)\in V(G)\times V(G)} x^{d(u,v)}.$$

**Theorem 12.** ([11]) The generating function of the sequence of ordered Hosoya polynomials of $\Gamma_n$ is
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\[ f(x,z) = \sum_{n \geq 0} H(\Gamma_n, x) z^n = \frac{1 + z + xz(1 - z)}{1 - z - z^2 + xz(-1 - z + z^2)} \]

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