On borderenergetic and L-borderenergetic graphs

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Abstract. A graph $G$ of order $n$ is said to be borderenergetic if its energy is equal to $2n - 2$. In this paper, we study the borderenergetic and Laplacian borderenergetic graphs.

Keywords. energy (of graph), adjacency matrix, Laplacian matrix, signless Laplacian matrix.

1 Introduction

We first recall some definitions that will be kept throughout. Let $G$ be a simple graph with $n$ vertices and $m(G)$ edges, and $A(G)$ denotes its adjacency matrix. Let $L(G) = D(G) - A(G)$ and $Q(G) = D(G) + A(G)$ be the Laplacian and signless Laplacian matrix of the graph $G$, respectively, where $D(G) = [d_{ij}]$ is the diagonal matrix whose entries are the degree of vertices, i.e., $d_{ii} = \deg(v_i)$ and $d_{ij} = 0$ for $i \neq j$.

The energy of $G$ is a graph invariant which was introduced by Ivan Gutman [6]. It is defined as $E(G) = \sum_{i=1}^{n} |\lambda_i|$, where $\lambda_i$'s are eigenvalues of $G$. If $0 = \mu_1 \leq \mu_2 \leq \cdots \leq \mu_{n-1} \leq \mu_n$ and $q_1 \leq q_2 \leq \cdots \leq q_{n-1} \leq q_n$ are the Laplacian and signless Laplacian eigenvalues of $G$ then the quantities $E_L(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m(G)}{n}|$ and $E_Q(G) = \sum_{i=1}^{n} |q_i - \frac{2m(G)}{n}|$ are called the Laplacian and signless Laplacian energy of $G$, respectively. Details on the properties of Laplacian and signless Laplacian energy can be found in [6,8,13].

The first borderenergetic graph was discovered by Hou et al. in 2001 [11], but in that time it did not attract much attention. Recently, Gong et al in [5] studied the graphs with the same

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energy as a complete graph. They put forward the concept of borderenergetic graphs.

A graph $G$ on $n$ vertices is said to be borderenergetic if its energy equals the energy of the complete graph $K_n$, i.e., if $E(G) = E(K_n) = 2(n - 1)$. In [5], it was shown that there exist borderenergetic graphs on order $n$ for each integer $n \geq 7$. The number of borderenergetic graphs were determined for $n = 7, 8, 9$ [5], $n = 10, 11$ [14, 12] and $n = 12$ [4].

In [12], a family of non-regular and non-integral borderenergetic threshold graphs was discovered. In [3], the authors obtained three asymptotically tight bounds on the number of edges of borderenergetic graphs. We refer the readers to [2, 15] for more information.

An analogous concept as borderenergetic graphs, called Laplacian borderenergetic graphs was proposed in [19]. That is, a graph $G$ of order $n$ is Laplacian borderenergetic or $L$-borderenergetic for short, if $E_L(G) = E_L(K_n) = 2n - 2$.

In [11], Deng et al. presented some asymptotically bounds on the order and size of $L$-borderenergetic graphs. Also, they showed that all trees, cycles, the complete bipartite graphs, and many 2-connected graphs are not $L$-borderenergetic. They showed in [2], a kind of threshold graphs are $L$-borderenergetic.

Lu et al. in [16] presented all non-complete $L$-borderenergetic graphs of order $4 \leq n \leq 7$ and they constructed one connected non-complete $L$-borderenergetic graph on $n$ vertices for each integer $n \geq 4$, which extends the result in [20] and completely confirms the existence of non-complete $L$-borderenergetic graphs. Particularly, they proved that there are at least $\frac{n}{2} + 4$ non-complete $L$-borderenergetic graphs of order $n$ for any even integer $n \geq 6$.

Hakimi-Nezhaad et al. in [10] generalized the concept of borderenergetic graphs for the signless Laplacian matrices of graphs. That is, a graph $G$ of order $n$ is signless Laplacian borderenergetic or $Q$-borderenergetic for short, if $E_Q(G) = E_Q(K_n) = 2n - 2$. Also, they constructed sequences of Laplacian borderenergetic non-complete graphs by means of graph operations, and all the non-complete and pairwise non-isomorphic $L$-borderenergetic and $Q$-borderenergetic graphs of small order $n$ are depicted for $n$ with $4 \leq n \leq 9$, see Appendix.

Tao et al. in [18] considered the extremal number of edges of non-complete $L$-borderenergetic graph, then use a computer search to find out all the $L$-borderenergetic graphs on no more than 10 vertices.

**Main Results**

Here, we present some basic theorem used to study borderenergetic and $L$-borderenergetic and $Q$-borderenergetic graphs.

**Theorem 1.1.** We have the following statements:

1) [5]. There are no borderenergetic graphs of order $n \leq 6$. 

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2) [5]. There exists a unique borderenergetic graph of order 7.

3) [5]. For any \( n \geq 7 \), there exist borderenergetic graphs of order \( n \).

4) [5]. There are exactly 6 borderenergetic graphs of order 8.

5) [5]. There are exactly 17 borderenergetic graphs of order 9.

6) [14, 17]. There are exactly 49 borderenergetic graphs of order 10.

7) [17]. There are exactly 158 borderenergetic graphs of order 11, of which 157 are connected.

8) [3]. There are exactly 572 connected borderenergetic graphs of order 12.

8) [5]. For each integer \( n \) (\( n \geq 13 \)), there exists a non-complete borderenergetic graph of order \( n \).

**Theorem 1.2.** [9]. A borderenergetic graph of order \( n \) must possess at least \( 2n - 2 \) edges.

**Theorem 1.3.** [3]. Let \( G \) be a \( k \)-regular integral graph of order \( n \) with \( t \) non-negative eigenvalues. If \( E(G) = 2(n - t + k) \) then \( E(\tilde{G}) = 2(n - 1) \), where \( \tilde{G} \) is complement of graph \( G \).

**Theorem 1.4.** [15].

1) There is no non-complete borderenergetic graph with maximum degree \( \triangle = 2 \) or 3.

2) Let \( G \) be a non-complete borderenergetic graph of order \( n \) with maximum degree \( \triangle = 4 \). Then \( G \) must have the following properties:
   (i) \( e(G) = 2n \) or \( 2n - 1 \);
   (ii) \( |G| \leq 21 \);
   (iii) \( G \) is non-bipartite;
   (iv) the nullity, i.e., the multiplicity of eigenvalue 0, of \( G \) is 0.

3) Let \( G \) be a \( 4 \)-regular non-complete borderenergetic graph of order \( n \) and \( H \) is a maximal bipartite subgraph of \( G \). Then \( m(G) - m(H) \geq 3 \).

**Theorem 1.5.** [15]. No borderenergetic graphs have minimum degree \( n - 2 \). Besides, for each integer \( n \geq 7 \), there exists a connected noncomplete borderenergetic graph of order \( n \) with minimum degree \( n - 3 \) and for each even integer \( n \geq 8 \), there exists a noncomplete borderenergetic graph of order \( n \) with minimum degree \( n - 4 \).

**Theorem 1.6.** We have the following statements:

1) [11]. There are exactly two non-complete \( L \)-borderenergetic disconnected graphs of orders 4 and 5, respectively.

2) [11]. There are exactly five non-complete \( L \)-borderenergetic disconnected graphs of order 6.
3) [10]. There are exactly five non-complete L-borderenergetic disconnected graphs of order 7.

4) [18]. There are totally 18 L-borderenergetic connected graphs on less than 8 vertices.

5) [10, 18]. There are exactly 31 L-borderenergetic connected graphs and 27 disconnected graphs of order 8.

6) [10, 18]. There are exactly 16 L-borderenergetic graphs and 26 disconnected graphs of order 9.

7) [10, 18]. There are exactly 120 L-borderenergetic connected graphs on 10 vertices.

Theorem 1.7. [10] There is no Laplacian borderenergetic tree with $n \geq 3$ vertices.

Theorem 1.8. [1]. If $G$ is a complete bipartite graph $K_{a,b}(1ab)$, then $G$ is not L-borderenergetic.

Theorem 1.9. [1]. If $G$ is a 2-connected graph with maximum degree $\Delta = 3$ and $t(G) \geq 7$ then $G$ is not L-borderenergetic, where $t(G)$ the number of vertices of degree 3 in $G$.

Theorem 1.10. [10].

1) There are no non-complete Q-borderenergetic graph of order $n \leq 5$ and 7.

2) There are exactly two non-complete Q-borderenergetic of order 6.

3) There exist exactly fourteen non-complete Q-borderenergetic graphs of order 8.

4) There exist exactly sixteen non-complete Q-borderenergetic graphs of order 9.

References


Appendix

Figure 1. All Laplacian borderenergetic dis-connected graphs of order 4 and 5.

Figure 2. All Laplacian borderenergetic dis-connected graphs of order 6.

Figure 3. All Laplacian borderenergetic dis-connected graphs of order 7.
Figure 4. All Laplacian borderenergetic dis-connected graphs of order 8.
Figure 5. All Laplacian borderenergetic dis-connected graphs of order 9.