

## Energy of graphs

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**Abstract.** Let  $G = (V, E)$  be a simple graph of order  $n$  and  $m$  edges. The energy of a graph  $G$ , denoted by  $\mathcal{E}(G)$ , is defined by Ivan Gutman as the sum of the absolute values of all eigenvalues of  $G$ . In the past decade, interest in graph energy has increased and many different versions have been introduced. The aim of this paper is to present the graph energy of normal Cayley graphs in terms of their character tables.

**Keywords.** energy of graph, Laplacian energy, characteristic polynomial.

### 1 Introduction

The energy  $\mathcal{E}(G)$  of a graph  $G$  is equal to the sum of the absolute values of the graph eigenvalues, namely the sum of the eigenvalues of the adjacency matrix  $A(G)$  of  $G$ . The origin of this concept comes from the  $\pi$ -electron energy in the Hückel molecular orbital model, but has also gained purely mathematical interest. In the past decade many kinds of energy have been introduced. In 2006, Gutman and Zhou defined the Laplacian energy of a graph as the sum of the absolute deviations of the eigenvalues of its Laplacian matrix [15]. The signless Laplacian, the distance, the incidence and many other versions of energy associated with a graph were defined, see [23]. In 2010, Cavers, Fallat and Kirkland first studied the Normalized Laplacian energy of a graph known as the Randić energy related to the Randić index [3].

Let  $G = (V, E)$  be a simple graph on  $n$  vertices and  $m$  edges with vertex set  $V(G) = \{v_1, v_2, \dots, v_n\}$ . If vertices  $v_i$  and  $v_j$  are adjacent, we denote that by  $v_i v_j \in E(G)$ . The adjacency matrix  $A(G)$  of  $G$  is defined as follows: the entry  $a_{ij} = 1$  if  $v_i v_j \in E(G)$  and 0 otherwise.

The characteristic polynomial  $\chi(X)$  of graph  $X$  with adjacency matrix  $A$  is defined as  $\chi(X) = \det(xI - A)$ . It is a monic polynomial of degree  $n$ . The roots of the characteristic polynomial are eigenvalues of  $X$  and form the spectrum of  $X$ . Since all considered graphs are undirected, the adjacency matrix  $A$  is symmetric. Consequently, all eigenvalues are real.

Suppose  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$  denote the eigenvalues of  $A(G)$ . Some well-known results concerning the energy of graph are as follows:

$$\sum_{i=1}^n \lambda_i = 0, \tag{1}$$

$$\sum_{i=1}^n \lambda_i^2 = 2m, \tag{2}$$

$$\det(A) = \prod_{i=1}^n \lambda_i. \tag{3}$$

The energy of the graph  $G$  is defined as

$$\mathcal{E}(G) = \sum_{i=1}^n |\lambda_i|,$$

where  $\lambda_i, i = 1, 2, \dots, n$  are the eigenvalues of graph  $G$ , see [1–7, 12–14, 18] as well as [8–11].

## 2 Definitions and Preliminaries

A general linear group  $GL(\mathcal{V})$  of vector space  $\mathcal{V}$  is the set of all  $A \in \text{End}(\mathcal{V})$  where  $A$  is invertible. A representation of a group  $\Gamma$  is a homomorphism  $\alpha : \Gamma \rightarrow GL(\mathcal{V})$  and the degree of  $\alpha$  is equal to the dimension of  $\mathcal{V}$ . A trivial representation is a homomorphism  $\alpha : \Gamma \rightarrow \mathbb{C}^*$  where  $\alpha(g) = 1$  for all  $g \in \Gamma$ . Let  $\varphi : \Gamma \rightarrow GL(\mathcal{V})$  be a representation with  $\varphi(g) = \varphi_g$ , the character  $\chi_\varphi : \Gamma \rightarrow \mathbb{C}$  of  $\varphi$  is defined as  $\chi_\varphi(g) = \text{tr}(\varphi_g)$ . An irreducible character is the character of an irreducible representation and the character  $\chi$  is linear, if  $\chi(1) = 1$ . We denote the set of all irreducible characters of  $G$  by  $\text{Irr}(\Gamma)$ , see [17] for more details on character theory.

Let  $\Gamma$  be a finite group with symmetric subset  $S$ . The symmetric subset  $S$  is a normal if  $g^{-1}Sg = S$ , for all  $g \in \Gamma$ . The Cayley graph  $X = \text{Cay}(\Gamma, S)$  has the elements of  $\Gamma$  as its vertices and two vertices  $x$  and  $y$  are adjacent if and only if  $yx^{-1} \in S$ .

**Theorem 2.1.** *Let  $\Gamma$  be a finite group with a normal symmetric subset  $S$ . Let  $A$  be the adjacency matrix of the graph  $X = \text{Cay}(\Gamma, S)$ . Then the eigenvalues of  $A$  are given by  $[\lambda_\varphi]^{\varphi(1)^2}$ , where*

$$\lambda_\varphi = \frac{1}{\varphi(1)} \sum_{s \in S} \varphi(s)$$

and  $\varphi \in \text{Irr}(\Gamma)$ .

**Example 2.2.** [19] Consider the dihedral group  $D_8$  with the following presentation

$$D_8 = \langle a, b : a^4 = b^2 = 1, b^{-1}ab = a^{-1} \rangle.$$

The character table of dihedral group  $D_8$  is reported in Table 1. Let  $S = \{a, a^{-1}\}$ , then by using Theorem 2. 1, all eigenvalues of  $D_8$  are:

$$\lambda_{\chi_1} = \lambda_{\chi_3} = 2, \lambda_{\chi_2} = \lambda_{\chi_4} = -2 \text{ and } \lambda_{\chi_5} = 0.$$

In other words, suppose  $S = \{a, a^{-1}\}$ , then the spectrum of  $X = \text{Cay}(D_8, S)$  is as follows:

$$\text{Spec}(X) = \{[-2]^2, [0], [2]^2\}.$$

$\mathcal{M}(D_8)$	1	$g_1$	$g_2$	$g_3$	$g_4$
$\chi_1$	1	1	1	1	1
$\chi_2$	1	-1	1	1	-1
$\chi_3$	1	1	-1	1	-1
$\chi_4$	1	-1	-1	1	1
$\psi_5$	2	0	0	-2	0

**Table 1.** The character table of group  $D_8$ .

### 3 Main Results

Suppose  $\Gamma$  is a group with irreducible characters  $\text{Irr}(\Gamma)$  and a normal symmetric subset  $S$ . Then the energy of Cayley graph  $X = \text{Cay}(\Gamma, S)$  is

$$\mathcal{E}(X) = \sum_{\varphi \in \text{Irr}(\Gamma)} \varphi(1) \left| \sum_{s \in S} \varphi(s) \right|.$$

Indeed, if  $\Gamma$  is abelian group then it is a well-known fact that all characters are linear and so

$$\mathcal{E}(X) = \sum_{\varphi \in \text{Irr}(\Gamma)} \left| \sum_{s \in S} \varphi(s) \right|.$$

**Corollary 3.1.** Suppose  $\Gamma$  is a finite group with exactly  $n$  conjugacy classes  $\{x_1^\Gamma, \dots, x_n^\Gamma\}$  and  $X = \text{Cay}(\Gamma, S)$ . Then

$$\mathcal{E}(X) = \sum_{\varphi \in \text{Irr}(\Gamma)} \varphi(1) \sum_{i=1}^n |x_i^\Gamma| \times \left| \sum_{s \in x_i^\Gamma} \varphi(s) \right|.$$

**Theorem 3.2.** Suppose  $\Gamma$  is a finite group with exactly  $n$  conjugacy classes and  $S, S'$  are normal subsets of  $\Gamma \setminus \{e\}$  such that  $S' = (\Gamma \setminus S) \cup \{e\}$ . Moreover, we assume that  $\Gamma = \text{Cay}(\Gamma, S)$ ,  $\Gamma' = \text{Cay}(\Gamma, S')$ . Set  $\text{Spec}(\Gamma) = \{\lambda_1, \dots, \lambda_n\}$  and  $\text{Spec}(\Gamma') = \{\beta_1, \dots, \beta_n\}$ . Then  $\beta_i = -\lambda_i - 1$ ,  $1 \leq i \leq n$ .

*Proof.* It is clear that  $S'$  is a symmetric and generating subset of  $G$ . So,

$$\begin{aligned} \beta_i &= \frac{1}{\chi_i(e)} \sum_{s' \in S'} \chi_i(s') \\ &= \frac{1}{\chi_i(e)} \left( \sum_{s' \in \Gamma} \chi_i(s') - \sum_{s \in S \cup \{e\}} \chi_i(s) \right) \\ &= \frac{1}{\chi_i(e)} \sum_{s' \in \Gamma} \chi_i(s') - \frac{1}{\chi_i(e)} \sum_{s \in S} \chi_i(s) - \frac{1}{\chi_i(e)} \chi_i(e) \\ &= -\lambda_i - 1, \end{aligned}$$

proving the result. □ □

**Theorem 3.3.** Suppose  $\Gamma = \{g_1, g_2, \dots, g_n\}$  and  $F$  is a subset of  $\{1, 2, \dots, n\}$  such that for all  $j \in F$ , the order of  $g_j$  is a power of a prime  $p_j$ . Define  $S = \cup_{j \in F} g_j^\Gamma$  and  $X = \text{Cay}(\Gamma, S)$ . We also assume that for each  $\chi \in \text{Irr}(\Gamma)$ ,  $\chi(g_j)$  is an integer. Then  $\text{Spec}(X) = \{\lambda_1, \dots, \lambda_n\}$ , where  $\lambda_i = \sum_{j \in F} |g_j^\Gamma| \left(1 + \frac{k_j p_j}{\chi_i(e)}\right)$ , for some integer  $k_j$ .

**Proof.** Suppose  $\chi_l$ ,  $1 \leq l \leq |\text{Irr}(\Gamma)|$ , is an arbitrary irreducible character of  $\Gamma$ . Then by [17, Corollary 22.27],  $\chi_l(g_j) \equiv \chi_l(e) \pmod{p_j}$ . Thus,  $\chi_l(g_j) - \chi_l(e) = p_j k_j$ , for some integer  $k_j$ . This implies that  $\frac{\chi_l(g_j)}{\chi_l(e)} = 1 + \frac{p_j k_j}{\chi_l(e)}$  and so,

$$\lambda_l = \sum_{s \in S} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j \in F} \sum_{s \in g_j^\Gamma} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j \in F} \sum_{s \in g_j^\Gamma} \left(1 + \frac{p_j k_j}{\chi_l(e)}\right) = \sum_{j \in F} |g_j^\Gamma| \left(1 + \frac{p_j k_j}{\chi_l(e)}\right).$$

**Corollary 3.4.** Suppose  $\Gamma$  is a finite group,  $S$  is a normal subset of  $\Gamma$  such that all elements of  $S$  are involutions and  $X = \text{Cay}(\Gamma, S)$ . If  $F$  is a representative set for  $\Gamma$ -conjugacy classes of  $S$  then the following statements hold:

1. if  $\Gamma$  is abelian then all  $X$ -eigenvalues are odd or all of them are even,
2. if  $\Gamma$  is a simple group then  $\lambda_\chi = \sum_{g \in F} |g^\Gamma| \left(1 + \frac{4k_g}{\chi(e)}\right)$ .

The energy of Cayley graphs of cyclic, dicyclic, semi-dihedral, Suzuki, Ree groups are computed in [16].

**Example 3.5.** Let  $A_5$  be the alternating group on five symbols. In [19], it is proved that the conjugacy classes of alternating group  $A_5$  are

$$\langle \rangle^\Gamma, (1,2)(3,4)^\Gamma, (1,2,3)^\Gamma, (1,2,3,4,5)^\Gamma, (1,2,3,5,4)^\Gamma,$$

where  $a = (1,2,3,4,5)$  and  $b = (2,3)(4,5)$  are the generators of this group. Let  $S = \{a, a^{-1}, b\}$ . Then the Cayley graph  $X = \text{Cay}(A_5, S)$  is as depicted in Figure 1. The spectrum of this graph is

$$3^1, 2.7^3, 2.3^5, 1.8^3, 1.6^4, 1^9, 0.6^5, -0.1^3, -0.4^4, -1.3^4, -1.4^3, -1.6^5, -2^4, -2.57^4, -2.6^3.$$

Hence,  $\mathcal{E}(\text{Cay}(A_5, S)) \cong 91.78$ .

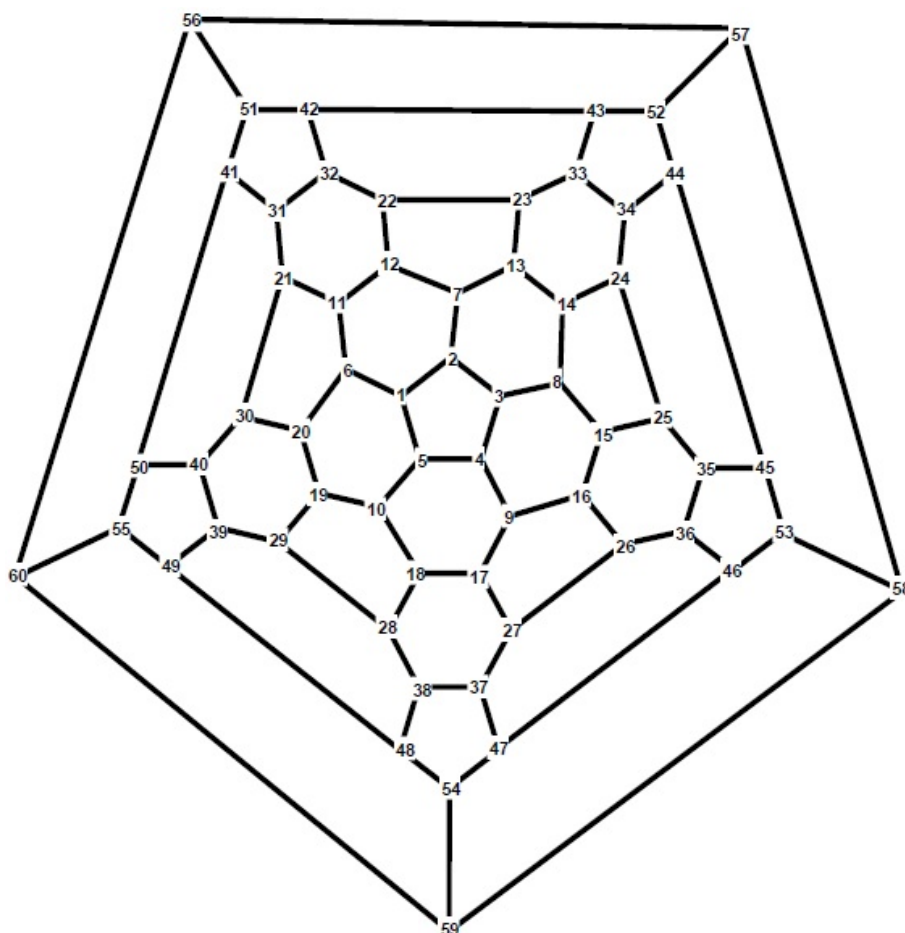


Figure 1. Cayley graph  $X = \text{Cay}(A_5, S)$ , where  $S = \{()1, 2, 3, 4, 5), (1, 5, 4, 3, 2), (2, 3)(4, 5)\}$ .

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