On the edge energy of some specific graphs

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Academic Editor: Ivan Gutman

Abstract. Let $G = (V,E)$ be a simple graph. The energy of $G$ is the sum of absolute values of the eigenvalues of its adjacency matrix $A(G)$. In this paper we consider the edge energy of $G$ (or energy of line of $G$) which is defined as the absolute values of eigenvalues of edge adjacency matrix of $G$. We study the edge energy of specific graphs.

Keywords. energy, edge energy, edge adjacency matrix, line graph.

1 Introduction

In this paper, we are concerned with simple finite graphs, without directed, multiple, or weighted edges, and without self-loops. Let $G$ be such a graph, with vertex set $V(G) = \{v_1,v_2,...,v_n\}$. Let $A(G)$ be the $(0,1)$-adjacency matrix of graph $G$. The characteristic polynomial of $G$ is $\det(A(G) - \lambda I)$ and is denoted by $P_G(\lambda)$. The roots of $P_G(\lambda)$ are called the adjacency eigenvalues of $G$ and since $A(G)$ is real and symmetric, the eigenvalues are real numbers. If $G$ has $n$ vertices, then it has $n$ eigenvalues and we denote its eigenvalues in descending order $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $\lambda_1, \lambda_2, ..., \lambda_s$ be the distinct eigenvalues of $G$ with multiplicity $m_1, m_2, ..., m_s$, respectively. The multiset $\text{Spec}(G) = \{(\lambda_1)^{m_1},(\lambda_2)^{m_2},..., (\lambda_s)^{m_s}\}$ of eigenvalues of $A(G)$ is called the adjacency spectrum of $G$. The energy $E(G)$ of the graph $G$ is defined as the sum of the absolute values of its eigenvalues

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$
Details and more information on graph energy can be found in [11,15–17,23,24]. There are many kinds of graph energies, such as incidence energy [3,5], Laplacian energy [10], matching energy [8,19,20] and Randić energy [2,4,22].

The line graph of $G$ is denoted by $L(G)$, the basic properties of line graphs are found in textbooks, e.g., in [18]. The iterated line graphs of $G$ are then defined recursively as $L^2(G) = L(L(G)), L^3(G) = L(L^2(G)), \ldots, L^k(G) = L(L^{k-1}(G))$. The basic properties of iterated line graph sequences are summarized in the articles [6,7]. Authors in [21] have shown that, if $G$ is a regular graph of order $n$ and of degree $r \geq 3$, then for each $k \geq 2$, $E(L^k(G))$ depends solely on $n$ and $r$. In particular, $E(L^2(G)) = 2nr(r - 2)$. In [14] authors has established relations between the energy of the line graph of a graph $G$ and the energies associated with the Laplacian and signless Laplacian matrices of $G$.

In this paper we consider the edge energy of a graph (energy of line graph) and compute it for some specific graphs.

2 Main results

In this section we consider the edge energy of a graph (or the energy of the line of a graph) and obtain some of its properties. First we recall the definition of the edge adjacency matrix of a graph. Note that the edge adjacency energy of a graph is just the ordinary energy of the line graph and has studied in detail. For instance see [14,21].

Definition 1. Let $G$ be a connected graph with edge set $\{e_1, \ldots, e_m\}$. The edge adjacency matrix of $G$ is defined as a square matrix $A_e = A_e(G) = [a_{ij}]$ where $a_{ij} = 0$ if $i = j$ or $e_i$ and $e_j$ are not adjacent, and $a_{ij} = 1$ if edges $e_i$ and $e_j$ are adjacent.

This matrix is symmetric and all its eigenvalues are real. The edge characteristic polynomial of $G$ is $\phi_e(x) = \det(A_e - xI)$.

Definition 2. The edge energy of a graph $G$ is denoted by $E_e(G)$ and defined as

$$E_e(G) = \sum_{i=1}^{m} |\mu_i|,$$

where $\mu_1, \ldots, \mu_m$ are eigenvalues of $A_e(G)$.

Here we are interested to obtain edge energy of some specific graphs. First we consider star graphs $K_{1,n}$.

Theorem 2.1. For every natural $n$, $E_e(K_{1,n}) = E(K_n)$.

Proof. We know that the star graph $K_{1,n}$ has $n$ edges. All its edges are adjacent in a vertex (center). The edge adjacency matrix of this graph is

$$A_e(K_{1,n}) = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{pmatrix} = J - I,$$
where \( J \) is a square matrix whose all its arrays are 1, and \( I \) is identity matrix. The eigenvalues of this matrix are the eigenvalues of adjacency matrix of \( K_n \) (see [9]). Therefore \( E_e(K_{1,n}) = E(K_n) \).

**Theorem 2.2.** For every natural \( n \), \( E_e(P_n) = E(P_{n-1}) \).

**Proof.** The graph path with \( n \) vertices, has \( n - 1 \) edges and no cycles. Its edge adjacency matrix is

\[
A_e(P_n) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 0 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

This matrix is exactly the adjacency matrix of \( P_{n-1} \) (see [9]). Therefore we have the result.

**Theorem 2.3.** For every natural \( n \geq 3 \), \( E_e(C_n) = E(C_n) \).

**Proof.** The graph cycle with \( n \) vertices, has \( n \) edges. Its edge adjacency matrix is

\[
A_e(C_n) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 \\
1 & 0 & 0 & \cdots & 0 & 1
\end{pmatrix}
\]

This matrix is exactly the adjacency matrix of \( C_n \) (see [9]) and so we have the result.

Now we shall obtain the edge energy of some another graphs. Here we investigate the complete bipartite graphs \( K_{m,n} \).

**Lemma 2.4.** (i) The edge characteristic polynomial of \( K_{m,n} \) is

\[
(x + 2)^{(m-1)(n-1)}(x - (m + n - 2))(x + 2 - n)^{m-1}(x + 2 - m)^{n-1}.
\]

(ii) \( E_e(K_{m,n}) = 4(m - 1)(n - 1) \).

**Proof.** (i) We can see that the edge adjacency matrix of \( K_{m,n} \) is \( mn \times mn \) matrix

\[
A_e(K_{m,n}) = \begin{pmatrix}
I_n - I_n & I_n & I_n & \cdots & I_n \\
I_n & I_n - I_n & I_n & \cdots & I_n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
I_n & I_n & I_n & \cdots & I_n - I_n
\end{pmatrix}.
\]
With simple computation,
\[
\phi_e(x) = \det(A_e(K_{m,n}) - xI) = (x + 2)^{(m-1)(n-1)}(x - (m + n - 2))(x + 2 - n)^{m-1}(x + 2 - m)^{n-1}.
\]

(ii) It follows from Part (i).

Now, we consider two families of graphs and obtain their edge energy. The friendship (or Dutch-Windmill) graph \(F_n\) is a graph that can be constructed by coalescence \(n\) copies of the cycle graph \(C_3\) of length 3 with a common vertex. The Friendship theorem of Paul Erdős, Alfred Rényi and Vera T. Sós [12], states that graphs with the property that every two vertices have exactly one neighbour in common are exactly the friendship graphs. The Figure 1 shows some examples of friendship graphs. Let to obtain the energy of \(F_n\). First we need the following theorem:

**Theorem 2.5.** [11]

(i) The characteristic polynomial of \(F_n\) is

\[
P_{F_n}(x) = (x + 1)(x^2 - 1)^{n-1}(x^2 - x - 2n).
\]

(ii) The spectrum of friendship graph \(F_n\) is

\[
\text{Spec}(F_n) = \left\{ \left(\frac{1}{2} - \frac{1}{2}\sqrt{1 + 8n}\right)^1, \left(-1\right)^n, \left(1\right)^{n-1}, \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 8n}\right)^1 \right\}.
\]

The following corollary is an immediate consequence of Theorem 2.5.

**Corollary 2.6.** The energy of friendship graph \(F_n\) is

\[
E(F_n) = \sqrt{1 + 8n} + 2n - 1.
\]

To obtain the edge energy of friendship graphs, we consider two following matrices:

\[
A = \begin{pmatrix}
0 & 1 & 1 \\
1 & 0 & 1 \\
1 & 1 & 0
\end{pmatrix}
\]

Figure 1. Friendship graphs \(F_2, F_3, F_4\) and \(F_n\), respectively.
and

\[ B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}. \]

It is easy to see that the edge adjacency matrix of \( F_n \) is \( 3n \times 3n \) matrix in the following lemma:

**Lemma 2.7.** The edge adjacency matrix of friendship graph \( F_n \) is

\[ A_e(F_n) = \begin{pmatrix} A & B & B & \cdots & B \\ B & A & B & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & B & \cdots & A \end{pmatrix}. \]

**Theorem 2.8.**  
(i) The edge characteristic polynomial of \( F_n \) is

\[ (x^2 - (2n - 1)x - 2)(x - 1)^{n-1}(x + 2)^{n-1}(x + 1)^n. \]

(ii) The edge energy of friendship graphs is \( E_e(F_n) = 4n - 3 + \sqrt{(2n - 1)^2 + 8}. \)

*Proof.*  
(i) Using Lemma 2.7 and simple computation we have the result.

(ii) It follows from Part (i). \( \square \)

Let us to consider book graphs. The \( n \)-book graph \( B_n \) can be constructed by joining \( n \) copies of the cycle graph \( C_4 \) with a common edge \( \{u, v\} \), see Figure 2.

![Figure 2. The book graphs B3 and B4, respectively.](image)

The following theorem gives the edge characteristic polynomial and edge energy of book graphs.

**Theorem 2.9.**  
(i) The edge characteristic polynomial of \( B_n \) is

\[ x(x - (n - 1))(x - (n + 1))(x - 1)^{n-1}(x + 2)^{n-1}(x + 1)^{n-1}. \]

(ii) The edge energy of book graph is \( E_e(B_n) = 6n - 2. \)
Proof. (i) Consider the following $n \times (n+1)$ matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

It is easy to see that the edge adjacency matrix of $B_n$ is the following $(3n+1) \times (3n+1)$ matrix:

$$A_e(B_n) = \begin{pmatrix} J - I & A & 0 \\ A^t & 0 & A^t \\ 0 & A & J - I \end{pmatrix}$$

With simple computation, we see that

$$\phi_e(x) = \det(A_e(B_n) - xI) = x(x - (n - 1))(x - (n + 1))(x - 1)^{n-1}(x + 2)^n(x + 1)^{n-1}.$$

(ii) It follows from Part (i). $\square$

In the end of this paper, we present the edge adjacency matrix of two another kind of graphs. For this purpose, we need the following matrix

$$B = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

For two graphs $G = (V, E)$ and $H = (W, F)$, the corona $G \circ H$ is the graph arising from the disjoint union of $G$ with $|V|$ copies of $H$, by adding edges between the $i$th vertex of $G$ and all vertices of $i$th copy of $H$ ([13]).

It is not difficult to see that the edge adjacency matrices of wheel graphs $W_{n+1} = C_n + K_1$ and graphs $C_n \circ K_1$ are in the form stated in the following theorem.

**Theorem 2.10.** (i) The edge adjacency matrix of wheel graphs $W_{n+1}$ is the following $2n \times 2n$ matrix:

$$A_e(W_n) = \begin{pmatrix} A_e(C_n) & B \\ B^t & J - I \end{pmatrix}$$

(ii) The edge adjacency matrix of graphs $C_n \circ K_1$ is the following $2n \times 2n$ matrix:

$$A_e(C_n \circ K_1) = \begin{pmatrix} A_e(C_n) & B \\ B^t & 0 \end{pmatrix}$$
References

[18] F. Harary, Graph Theory, Addison-Wesley, Reading, 1969 (Chapter 8).