On the automorphism group of cubic polyhedral graphs

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Academic Editor: Modjtaba Ghorbani

Abstract. In the present paper, we introduce the automorphism group of cubic polyhedral graphs whose faces are triangles, quadrangles, pentagons and hexagons.

Keywords. polyhedral graph, automorphism group, fullerene.

1 Introduction

Carbon atoms can bond into very large molecules. Named fullerenes, after U.S. engineer Buckminster Fuller (1895–1983), these carbon molecules have the same symmetry as a soccer ball, as shown in Figure 1. They are popularly called buckyballs. The most important member of fullerene graphs is $C_{60}$ fullerene with exactly 60 carbon atoms. In general, a fullerene is a cubic planar graph having all faces 5- or 6-cycles, see Figure 2. Examples include the 20-vertex dodecahedral graph, 24-vertex generalized Petersen graph $GP(12,2)$ and graph on 26 vertices truncated icosahedral graph.

A classical fullerene or briefly a fullerene is a cubic three connected graph whose faces entirely composed of pentagons and hexagons and we denote it by a PH-fullerene, see [18, 19]. The non-classical fullerenes are composed of triangles and hexagons or quadrangles and hexagons and we denote them by TH-fullerene or SH-fullerene, respectively. For see some problems concerning with fullerene graphs and many properties of them are derived, we refer the readers to [1, 2, 4, 7–9] as well as [11, 13, 16, 17, 20]. Fullerenes are special cases of

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DOI: 10.22061/JMNS.2017.511
a larger class of graphs, namely polyhedral graphs. A polyhedral graph is a three connected simple planar graph and in this paper, we consider only the cubic polyhedral graphs whose faces are a combination of triangles, quadrangles, pentagons and hexagons, see [4, 6].

An automorphism of graph \( X = (V, E) \) is a bijection \( \beta \) on \( V \) which preserves the edge set \( E \). In other words, \( e = uv \) is an edge of \( E \) if and only if \( e^\beta = u^\beta v^\beta \) is an edge of \( E \). Here, the image of vertex \( u \) is denoted by \( u^\beta \). The set of all automorphisms of graph \( X \) with the operation of composition is a group on \( V(X) \) denoted by \( Aut(X) \). Frucht [12] was the first who dealt with graph automorphism. Also quantitative measures based on graph automorphism have been developed, see [3].

Cubic polyhedral graph with \( t \) triangular, \( s \) quadrilateral, \( p \) pentagonal and \( h \) hexagonal faces and no other faces is denoted by a \((t, s, p, h)\)-polyhedral or briefly a \((t, s, p)\)-polyhedral graph. By these notations, a SPH-polyhedral graph is a planar graph whose faces are quadrangles, pentagons and hexagons. Let \( m \) be the number of edges in a given SPH-polyhedral graph \( F \). In [11] Fowler and his co-authors showed that fullerenes are realizable within 28 point groups. In [21] Kutnar et al. proved that for any PH-fullerene graph \( F \), \( |Aut(F)| \) divides 120. The present authors in [14] proved that for given TH-fullerene \( F \), \( |Aut(F)| \) divides 24 and in [15] they proved that for given SH-polyhedral graph \( F \), \( |Aut(F)| \) divides 48. These results are given in the following theorem.

**Theorem 1.1.** We have

- the size of automorphism group of classical fullerenes divides 120 [21].
- the size of automorphism group of TSH-fullerenes divides 24, [15].
- the size of automorphism group of SPH-fullerenes divides 48, [21].

A TPH-polyhedral graph \( F \) is one whose faces are triangles, pentagons and hexagons. In this paper, we prove the following theorem.

**Theorem A.** Let \( F \) be a TPH-polyhedral graph. Then the automorphism group of \( F \) is a subgroup of a \( \{2, 3, 5\} \)-group. Moreover, the order of \( Aut(F) \) divides \( 2^2 \times 3 \).
2 Definitions and Preliminaries

Let $G$ be a group and $\Omega$ a non-empty set. An action of $G$ on $\Omega$ denoted by $(G|\Omega)$ induces a group homomorphism $\varphi$ from $G$ into the symmetric group $S_\Omega$ on $\Omega$, where $\varphi(g)^a = g^a \ (a \in \Omega)$. The orbit of an element $a \in \Omega$ is denoted by $a^G$ and it is defined as the set of all $a^g$ where $g \in G$. Size of $\Omega$ is called the degree of this action. The kernel of this action is denoted by $\text{Ker}\varphi$. An action is faithful if $\text{Ker}\varphi = \{1\}$. The stabilizer of element $a \in \Omega$ is defined as $G_a = \{g \in G | a^g = a\}$. Let $H = G_a$ then for $\alpha, \beta \in \Omega \ (\alpha \neq \beta)$, $H_\beta$ is denoted by $G_{a,\beta}$. The orbit-stabilizer theorem implies that $|a^G|.|G_a| = |G|$. For every $g \in G$, let $\text{fix}(g) = \{\alpha \in X | a^g = \alpha\}$, then we have:

Lemma 2.1. (Cauchy–Frobenius Lemma) Let $G$ acts on set $\Omega$, then the number of orbits of $G$ is

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix}(g)|.$$

Example 2.2. Consider the fullerene graph $F_{96}$ depicted in Figure 3. If $\alpha$ denotes the rotation of $F_{96}$ through an angle of $60^\circ$ around an axis through the midpoints of the front and back
faces, then the corresponding permutation is \( \alpha = (1, 2, 3, 4, 5, 6)(7, 10, 14, 17, 20, 24)(8, 11, 15, 18, 21, 25)(9, 12, 16, 19, 23, 26)(13, 50, 58, 66, 74, 62)(22, 71, 47, 39, 55, 63)(27, 28, 29, 30, 31, 32)(33, 48, 56, 72, 64, 40)(34, 49, 57, 73, 65, 41)(35, 51, 59, 75, 67, 43)(36, 52, 60, 76, 68, 44)(37, 53, 61, 77, 69, 45)(38, 54, 62, 78, 70, 46)(79, 80, 81, 82, 83, 84)(85, 86, 87, 88, 89, 90)(91, 96, 95, 94, 93, 92). Thus, one of orbits of subgroup \( \langle \alpha \rangle \) containing the vertex 1 is \( 1^{\langle \alpha \rangle} = \{1, 2, 3, 4, 5, 6\} \). Now, consider the axis symmetry element which fixes vertices \( \{1, 4, 8, 18, 43, 44, 59, 60, 85, 88, 92, 95\} \), the corresponding permutation is \( \beta = (2, 6)(3, 5)(7, 9)(10, 26)(11, 25)(12, 24)(13, 71)(14, 23)(15, 21)(16, 20)(17, 19)(22, 50)(27, 28)(29, 32)(30, 31)(33, 70)(34, 69)(35, 67)(36, 68)(37, 65)(38, 64)(39, 66)(40, 46)(41, 45)(42, 47)(48, 78)(49, 77)(51, 75)(52, 76)(53, 73)(54, 72)(55, 74)(56, 62)(57, 61)(58, 63)(79, 80)(81, 84)(82, 83)(86, 90)(87, 89)(91, 93)(94, 96). Let \( G = Aut(F_{96}) \), clearly \( G \geq \langle \alpha, \beta \rangle \) and the orbit-stabilizer property implies that \( |G| = |1^{G_1}| \times |G_1| \). Any symmetry of the polyhedral graph \( F_{96} \) which fixes vertex 1 must also fixes the opposite vertex 4. By applying orbit-stabilizer property, we found that \( |G_1| = |4^{G_1}| \times |G_{1,4}| \). It is easy to prove that \( |G_{1,4}| = 2 \) and hence \( |G| = 12 \). On the other hand, \( |\langle \alpha, \beta \rangle| = 12 \), where \( \alpha^4 = \beta^2 = 1, \beta^{-1}a\beta = \alpha^{-1} \). This leads us to conclude that \( G = \langle \alpha, \beta \rangle \cong D_{12} \).

![Figure 3. Labeling of fullerene \( F_{96} \).](image)

**Example 2.3.** Here, we compute the order of automorphism group of polyhedral graph \( F_{48} \)
depicted in Figure 4. Similar to the last example, if $\alpha$ denotes the rotation of $F_{48}$ through an angle of $90^\circ$ around an axis through the midpoints of the front and back faces, then the corresponding permutation is $\alpha = (1, 3, 5, 7)(2, 4, 6, 8)(9, 15, 26, 21)(10, 16, 27, 32)(11, 17, 28, 22)(12, 29, 23)(13, 19, 30, 24)(14, 20, 31, 25)(33, 45, 41, 37)(34, 46, 42, 38)(35, 47, 43, 39)(36, 48, 44, 40)$. Thus $1^{(\alpha)} = \{1, 3, 5, 7\}$ and consider the axis symmetry element which fixes no vertices: $\beta = (1, 2)(3, 8)(4, 7)(5, 6)(9, 20)(10, 19)(11, 18)(12, 17)(13, 16)(14, 15)(21, 31)(22, 29)(23, 28)(24, 27)(25, 26)(30, 32)(33, 34, 39)(35, 38)(36, 37)(41, 48)(42, 47)(43, 46)(44, 45)$.

If $G = \text{Aut}(F_{48})$, then $|G| = 2^{|G_2|}$ while no element fixes 2. This means that $|G_2| = 1$ and so $|G| = 2^{|G_2|}$. It is clear that $2^{\langle \alpha, \beta \rangle} = \{1, 2, 3, 4, 5, 6, 7, 8\}$ and thus $|2^{\langle \alpha, \beta \rangle}| = 8$. Similar to the last example, we can see that $|\langle \alpha, \beta \rangle| = 8$, where $\alpha^4 = \beta^2 = 1$ and $\beta^{-1}\alpha\beta = \alpha^{-1}$, hence $\text{Aut}(F_{48})$ is isomorphic with dihedral group $D_8$.

**Figure 4.** Labeling of fullerene $F_{48}$.

3 Main Results

**Lemma 3.1.** Let $F$ be a TPH-polyhedral graph, with automorphism group $\text{Aut}(F)$. Then the stabilizer $\text{Aut}(F)(u,v,w)$ of 2-arc $(u,v,w)$ is trivial.

**Proof.** It is similar to the proof of [21, Lemma 2].

**Proposition 3.2.** Let $F$ be a TPH-polyhedral graph with automorphism group $\text{Aut}(F)$ and $u \in V(F)$. Then the stabilizer $\text{Aut}(F)_u$ of $u$ is trivial or it is isomorphic to one of three groups: the cyclic group $Z_2$, the cyclic group $Z_3$ and the symmetric group $S_3$.

**Proof.** It is similar to the proof of [21, Lemma 2].
Proof of Theorem A. Let $F$ be a TPH-polyhedral graph with a non-trivial automorphism group, $T(F)$ be the set of triangles of $F$ and $P(F)$ be the set of pentagons of $F$. Let $A = Aut(F)$ and $t$, $p$ be the number of triangles and pentagons, respectively. We can see that

$$|G| = |K_O| \times |(G/K_O)_T| \times |O| = 2^a.3^b.|O|,$$

and so

$$|G| = 2^a.3^b.5^\gamma.|O'|.$$

We distinguish the following cases:

Case 1. $t = 1$ and $p = 9$. We claim that $|Aut(F)|$ divides $3 \times 2$. Suppose $2^2$ divides $|A|$ and $Syl_2(A)$ is of order $2^2$. The order of orbits of $T(F)$ is 1. By orbit-stabilizer theorem, we have $|K_T| = 2^2$, a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|K_T| = 3^2$, a contradiction.

Case 2. $t = 2$ and $p = 6$, we show that $|Aut(F)|$ divides $3 \times 2^2$. Suppose $2^3$ divides $|A|$ and $Syl_2(A)$ is of order $2^3$. Let $Syl_2(A)$ acts on triangles, then for an orbit $O$ of this action, we have $|O| = 1$ or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, a contradiction and if $|O| = 2$, then $|K_T| = 2^2$, a contradiction. Let $|Syl_5(A)| = 5$, then $|K_T| = 5$, a contradiction. If $|Syl_3(A)| = 3^2$, then we have $|O| = 1$ and so $|K_T| = 3^2$, a contradiction.

Case 3. $t = 3$ and $p = 3$, we prove that $|Aut(F)|$ divides $3 \times 2^2$. Let $2^3$ divides $|A|$ and $Syl_2(A)$ be of order $2^3$ acting on the set of triangles $T$. Hence, $|O| = 1$ or 2, similar to the last discussion, both of them are contradictions. Also, $|Syl_5(A)| = 5$ is a contradiction. If $|Syl_3(A)| = 3^2$, then the orbits of the action $Syl_3(A)$ on $P(F)$ are of order 3 and so $|K_T| = 3$, a contradiction.

It should be noted that in a given polyhedral $F$, no two triangles are adjacent, since $F$ is three connected.

Theorem 3.3. Let $F$ be a TSH-polyhedral graph. Then $Aut(F)$ is a subgroup of a $\{2,3\}$-group. Moreover, the order of $Aut(F)$ divides $24$.

Proof. By using Euler’s theorem, if $s = 0$, then $t = 4$ and then $F$ be TH- fullerene. On the other hand, if $s = 6$, then $t = 0$ and $F$ is a SH-fullerene. Let $F$ be a TSH-polyhedral graph with at least one triangle and one square. We show that $|Aut(F)|$ divides $24$. First, we prove that the $Syl_2(F)$ is of order 8. Suppose on the contrary that $2^4$ divides $|A|$ and $Syl_2(A)$ is of order $2^4$. Let $Syl_2(A)$ acts on the set of triangles, clearly the order of an orbit of an this action is 1 or 2. By orbit-stabilizer theorem, if $|O| = 1$, then $|K_T| = 2^3$, a contradiction and if $|O| = 2$, then $|K_T| = 2^2$, a contradiction. Also $|Syl_5(A)| = 5$ yields a contradiction. If $|Syl_3(A)| = 3^2$, $|K_q| = 3^2$ or $|K_q| = 3$, then we have a contradiction. This completes the proof. □

In [6] the authors derived the list of allowed symmetry groups for each class they constructed the smallest polyhedron for each allowed symmetry. In other words, we have two following theorems, see [8,10,11].
Theorem 3.4. For the bifaced cubic polyhedra described by the triple \((t, s, p)\), the possible point groups and vertex counts of minimal examples are

i. \((t, s, p) = (4, 0, 0)\):
\[ \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, A_4, S_4. \]

ii. \((t, s, p) = (0, 6, 0)\):
\[ C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_8, \mathbb{Z}_2 \times \mathbb{Z}_2, \]
\[ \mathbb{Z}_2 \times S_3, D_{12}, D_6, \mathbb{Z}_2 \times D_{12}, \mathbb{Z}_2 \times D_6, \mathbb{Z}_2 \times \mathbb{Z}_2 \times D_6. \]

iii. \((t, s, p) = (0, 0, 12)\):
\[ C_1, \mathbb{Z}_2, A_3, S_3, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times \mathbb{Z}_2, D_{12}, \mathbb{Z}_2 \times S_3, \]
\[ A_4, D_{20}, \mathbb{Z}_2 \times D_{12}, D_{24}, S_4, A_4 \times \mathbb{Z}_2, A_5, \mathbb{Z}_2 \times A_5. \]

Theorem 3.5. For cubic polyhedra with at least two face sizes chosen from \(\{3, 4, 5\}\) and no face of size greater than 6 described by the triple \((t, s, p)\), the possible point groups and vertex counts of minimal examples are

i. \((t, s, p) = (3, 1, 1)\):
\[ C_1, \mathbb{Z}_2. \]

ii. \((t, s, p) = (3, 0, 3)\):
\[ C_1, \mathbb{Z}_2, A_3, S_3, \mathbb{Z}_2 \times \mathbb{Z}_2. \]

iii. \((t, s, p) = (2, 3, 0)\):
\[ C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, D_{12}. \]

iv. \((t, s, p) = (2, 2, 2)\):
\[ C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2. \]

v. \((t, s, p) = (2, 1, 4)\):
\[ C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2. \]

vi. \((t, s, p) = (2, 0, 6)\):
\[ C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, S_3, \mathbb{Z}_2 \times S_3, D_{12}. \]

vii. \((t, s, p) = (1, 4, 1)\):
\[ C_1, \mathbb{Z}_2. \]

viii. \((t, s, p) = (1, 3, 3)\):
\[ C_1, \mathbb{Z}_2, A_3, S_3. \]

ix. \((t, s, p) = (1, 2, 5)\):
\[ C_1, \mathbb{Z}_2. \]

x. \((t, s, p) = (1, 1, 7)\):
\[ C_1, \mathbb{Z}_2. \]
xi. 
\( (t, s, p) = (1,0,9) : \)
\( C_1, \mathbb{Z}_2, A_3, S_3. \)

xii. 
\( (t, s, p) = (0,5,2) : \)
\( C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_5, D_{20}. \)

xiii. 
\( (t, s, p) = (0,4,4) : \)
\( C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, Z_4. \)

xiv. 
\( (t, s, p) = (0,3,6) : \)
\( C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, A_3, \mathbb{Z}_2 \times \mathbb{Z}_3, S_3, D_{12}. \)

xv. 
\( (t, s, p) = (0,2,8) : \)
\( C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2, D_8, \mathbb{Z}_2 \times D_8, D_{16}, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{Z}_2, Z_4. \)

xvi. 
\( (t, s, p) = (0,1,10) : \)
\( C_1, \mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2. \)

References